14.7 Extreme values and Saddle points

Local Extrema — 2nd derivative test.

$f_{xx}<0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ then $F(a,b)\to$ local max
$f_{xx}>0$ and $f_{xx}f_{yy}-f_{xy}^2>0$ then $F(a,b)\to$ local min
$f_{xx}f_{yy}-f_{xy}^2<0$ then $F(a,b)\to$ saddle point
$f_{xx}f_{yy}-f_{xy}^2=0$ then $F(a,b)\to$ inconclusive

1. Find Fx and Fy
2. Find point of critical point when $x=0$ and $y=0$. (solve y, substitute in x)
3. Find Fxx, Fyy, Fxy
4. Find D (discriminant)
5. Evaluate critical point of $F(x,y)$ to find value.

Absolute Extrema

1. Find Fx and Fy
2. Find point of critical point when $x=0$ and $y=0$. (solve y, substitute in x)
3. Sketch and mark segments with endpoints (L1 (0,2),(2,0))
4. table $x|y|F$ to record values and find abs. max and abs. min.

$\boxed{L1 \quad y=0 \quad 0\le x\le 4 \quad F(x,0)=x^2-2x(0)+4(0) = x^2 \quad F(0)=0,\ F(4)=16}$

$x^2-6x+12$ Quadratic expression! Find min value
take derivative, set to zero, solve for x.
evaluate at found point.

5. Evaluate critical point plug into original expression.

14.8 Lagrange Multipliers

Finding Extrema subject to a constraint

1. Find $fx=\lambda gx$ $fy=\lambda gy$ $fz=\lambda gz$ simultaneously!
2. Plug x,y,z values into constraint function $g(x,y,z)$.
3. Solve for λ.
4. Plug λ into x,y,z results from step 1 to get points.
5. Evaluate $F(x,y,z)$ at given points from step 4.

15.1 Double and iterated integrals over rectangles.

Fubini's Theorem for integral over region R.

$R=\{a\le x\le b, c\le y\le d\}$

$$\iint_R f(x,y)\,dA=\int_c^d\int_a^b f(x,y)\,dx\,dy=\int_a^b\int_c^d f(x,y)\,dy\,dx$$

$$\int_0^1\int_0^{10}\frac{yx^4}{y^2+1}\,dx\,dy \quad R:\ 0\le x\le 10,\ 0\le y\le 1$$

treating y as a constant

$$\frac{y}{y^2+1}\int_0^{10}x^4\,dx$$
$$\frac{y}{y^2+1}\cdot\left.\frac{x^5}{5}\right|_0^{10}$$
$$\frac{y}{y^2+1}\left(\frac{10^5}{5}-\frac{0^5}{5}\right)$$
$$\frac{y}{y^2+1}(2000)$$
$$\frac{2000y}{y^2+1}$$

$$\int_0^1\frac{2000y}{y^2+1}\,dy$$ let $u=y^2+1$, $du=2y\,dy$, $dy=\frac{1}{2y}du$

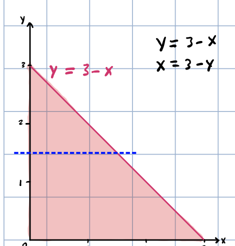
$$2000\int_0^1\frac{y}{y^2+1}\,dy$$
$$2000\int_1^2\frac{\cancel{y}}{u}\frac{1}{2\cancel{y}}\,du$$

Adjust integral boundaries
$y=0 \to u=1$
$y=1 \to u=2$

$$2000\int_1^2\frac{1}{2u}\,du$$
$$2000\cdot\frac{1}{2}\int_1^2\frac{1}{u}\,du = 1000(\ln 2-\cancel{\ln 1})$$ $\ln 1=0$
$$2000\cdot\frac{1}{2}\left[\ln u\right]_1^2 = 1000\ln 2$$

Double Integrals as volumes

$$V=\iint_R f(x,y)\,dA$$

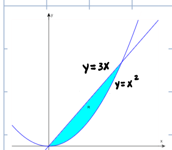


$y=3-x$
$x=3-y$

$$\int_0^3\int_0^{3-y}(45-5y^2)\,dx\,dy$$
$$\int_0^3\left[45x-5y^2x\right]_0^{3-y}dy$$
$$\int_0^3\left[45(3-y)-5y^2(3-y)\right]dy$$
$$\int_0^3\left[135-45y-15y^2+5y^3\right]dy$$
$$\int_0^3\left(5y^3-15y^2-45y+135\right)dy$$
$$\left[\frac{5}{4}y^4-5y^3-\frac{45}{2}y^2+135y\right]_0^3=\frac{675}{4}$$

15.2 Double integrals over general regions of R^2

$R=\{a\le x\le b,\ g(x)\le y\le h(x)\}$ $\quad \iint_R f(x,y)\,dA=\int_a^b\int_{g(x)}^{h(x)}f(x,y)\,dy\,dx$



1. Solve y equations to find point of intersection of both curves.

$y=3x$
$y=x^2$
$3x=x^2$
$x^2-3x=0$
$x(x-3)=0$
$0\le x\le 3$

$$\iint_R F(x,y)\,dA=\int_0^3\int_{x^2}^{3x}dy\,dx \quad \text{Vertical}$$

$R=\{c\le y\le d,\ g(y)\le x\le h(y)\}$ $\quad \iint_R f(x,y)\,dA=\int_c^d\int_{g(y)}^{h(y)}f(x,y)\,dx\,dy$

$y=3x$, $x=\frac{y}{3}$ $\quad y=x^2$, $x=\sqrt{y}$ $\quad y=3(3)$, $y=(3)^2=9$ $\quad y=3(0)$, $y=(0)^2=0$
$\frac{y}{3}\le x\le\sqrt{y}$

$$\iint_R F(x,y)\,dA=\int_0^9\int_{y/3}^{\sqrt{y}}dx\,dy \quad \text{Horizontal}$$

Reverse the order of integration

Imagine a vertical line passing through the graph from bottom to top. Identify the first y value it encounters and then the last. These are the new y-limits of integration.

Next identify the limits on x. Identify the lowest and then the highest value x can take. These are the new x-limits of integration.

$$\int_0^{\frac12}\int_0^{x^3}\cos(8\pi x^4)\,dy\,dx \qquad \frac{1}{32\pi}\int_0^{\frac{\pi}{2}}\cos(u)\,du$$
$u=8\pi x^4$, $du=32\pi x^3\,dx$, $dx=\frac{1}{32\pi x^3}du$

$$\int_0^{\frac12}\left[y\cos(8\pi x^4)\right]_0^{x^3}dx \qquad \frac{1}{32\pi}\left[\sin(u)\right]_0^{\frac{\pi}{2}}$$
$$\int_0^{\frac12}\left[x^3\cos(8\pi x^4)\right]dx \qquad \frac{1}{32\pi}\left(\sin\frac{\pi}{2}-\sin 0\right)$$

Adjust boundaries: $x=0\to u=0$, $x=\frac12\to u=\frac{\pi}{2}$

$$\frac{1}{32\pi}(1-0)$$
$$\int_0^{\frac{\pi}{2}}\cancel{x^3}\cos(u)\frac{1}{32\pi\cancel{x^3}}du \qquad \frac{1}{32\pi}(1)=\frac{1}{32\pi}$$
$$\int_0^{\frac{\pi}{2}}\frac{\cos(u)}{32\pi}du$$

15.3 Area by Double Integration

Given a curve and a line. Sketch. Use vertical/horizontal line test to find lower/upper x- and y-limits. Setup integral.

$$A=\iint_R dA$$

15.3 Average Value

Find the area of the region R.

$A(R)=\frac{\pi}{3}\cdot\frac{7\pi}{6}=\frac{7\pi^2}{18}$ $\qquad f_{avg}=\frac{1}{\text{area of }R}\iint_R f\,dA$

$$\frac{1}{\frac{7\pi^2}{18}}\int_0^{\frac{\pi}{3}}\int_0^{\frac{7\pi}{6}}\sin(x+y)\,dy\,dx$$
$$\frac{18}{7\pi^2}\int_0^{\frac{\pi}{3}}\left[-\cos(x+y)\right]_0^{\frac{7\pi}{6}}dx$$
$$\frac{18}{7\pi^2}\int_0^{\frac{\pi}{3}}\left[-\cos\left(x+\frac{7\pi}{6}\right)-(-\cos(x+0))\right]dx$$
$$\frac{18}{7\pi^2}\int_0^{\frac{\pi}{3}}\left(\cos x-\cos\left(x+\frac{7\pi}{6}\right)\right)dx$$
$$\frac{18}{7\pi^2}\left[\sin x\right]_0^{\frac{\pi}{3}}-\left[\sin\left(x+\frac{7\pi}{6}\right)\right]_0^{\frac{\pi}{3}}$$
$$\frac{18}{7\pi^2}\left[\sin\frac{\pi}{3}-\sin 0\right]-\left[\sin\left(\frac{\pi}{3}+\frac{7\pi}{6}\right)-\sin\left(0+\frac{7\pi}{6}\right)\right]$$
$$\frac{18}{7\pi^2}\left(\frac{\sqrt3}{2}\right)-\left(\sin\left(\frac{3}{2}\right)-\sin\left(\frac{7\pi}{6}\right)\right)$$
$$\frac{18}{7\pi^2}\left(\frac{\sqrt3}{2}\right)-\left(-1-\left(-\frac12\right)\right)$$
$$\frac{18}{7\pi^2}\left(\frac{\sqrt3}{2}\right)-\left(-\frac12\right)$$
$$\frac{18}{7\pi^2}\left(\frac{\sqrt3}{2}+\frac12\right)$$
$$\frac{18}{7\pi^2}\left(\frac{\sqrt3+1}{2}\right)$$
$$\frac{9(\sqrt3+1)}{7\pi^2}$$

15.4 Double Integrals in Polar Form

Change a Cartesian integral into an equivalent polar integral.

$$\iint_R f(x,y)\,dx\,dy=\iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \quad (r,\theta)$$

$x=r\cos\theta$, $y=r\sin\theta$, $r^2=x^2+y^2$, $\tan\theta=\frac{y}{x}$, $dx\,dy=r\,dr\,d\theta$

$$\int_0^{\ln 3}\int_0^{\sqrt{(\ln 3)^2-y^2}}e^{\sqrt{x^2+y^2}}\,dx\,dy \qquad 0\le y\le\ln 3 \text{ and } 0\le x\le\sqrt{(\ln 3)^2-y^2}$$

take $x^2=\sqrt{(\ln 3)^2-y^2}$
$x^2=(\ln 3)^2-y^2$
$x^2+y^2=(\ln 3)^2$
this is a circle of radius $\ln 3$.

use polar coordinates:
$x=r\cos\theta$, $y=r\sin\theta$
so, $x^2+y^2=r^2$ and $dx\,dy=r\,dr\,d\theta$
$D=0\le r\le\ln 3,\ 0\le\theta\le\frac{\pi}{2}$

$$\int_0^{\ln 3}\int_0^{\sqrt{(\ln 3)^2-y^2}}e^{\sqrt{x^2+y^2}}\,dx\,dy=\int_0^{\frac{\pi}{2}}\int_0^{\ln 3}e^r\,r\,dr\,d\theta$$

evaluate the integral — Integration by parts

$\int u v' = uv-\int u'v$
$u=r$, $v'=e^r$
$u'=1\,dr$, $v=\int e^r=e^r$
$\left[e^r r-\int e^r\,dr\right]_0^{\ln 3}$

$$\int_0^{\frac{\pi}{2}}\int_0^{\ln 3}e^r r\,dr\,d\theta$$
$$\int_0^{\frac{\pi}{2}}\left[e^r r-e^r\right]_0^{\ln 3}d\theta$$
$$\int_0^{\frac{\pi}{2}}\left(e^{\ln 3}\ln 3-e^{\ln 3}\right)-\left(e^0\cdot 0-e^0\right)d\theta$$
$$\int_0^{\frac{\pi}{2}}(3\ln 3-3+1)\,d\theta$$
$$\int_0^{\frac{\pi}{2}}(3\ln 3-2)\,d\theta$$
$$\left[3\theta\ln 3-2\theta\right]_0^{\frac{\pi}{2}}$$
$$\left[3\frac{\pi}{2}\ln 3-2\frac{\pi}{2}\right]_0^{\frac{\pi}{2}}$$
$$\frac{3\pi}{2}\ln 3-\pi$$

Change a polar integral into a Cartesian integral.

$x=r\cos\theta$, $r^2=x^2+y^2$, $\tan\theta=\frac{y}{x}$, $dx\,dy=r\,dr\,d\theta$
$y=r\sin\theta$,

$$\int_0^{\frac{\pi}{4}}\int_0^{5\sec\theta}r^5\sin^2\theta\,dr\,d\theta$$

Rewrite:
$$\int_0^{\frac{\pi}{4}}\int_0^{5\sec\theta}r^4\sin^2\theta\ r\,dr\,d\theta$$
$$\int_0^{\frac{\pi}{4}}\int_0^{5\sec\theta}r^2\cdot r^2\sin^2\theta\ r\,dr\,d\theta$$
$$\int_0^{\frac{\pi}{4}}\int_0^{5\sec\theta}r^2(r\sin\theta)^2\ r\,dr\,d\theta$$

since the limits of r are $r=0$ and $r=5\sec\theta$
$r=5\sec\theta$ $\quad \sec\theta=\frac{1}{\cos\theta}$
$r=\frac{5}{\cos\theta}$
$r\cos\theta=5$
$x=5$

since the limits of θ are: $\theta=0$ and $\theta=\frac{\pi}{4}$
$\theta=\frac{\pi}{4}$ $\quad \frac{y}{x}=\tan\frac{\pi}{4}$
$\tan\theta=\frac{y}{x}$ $\quad \frac{y}{x}=1$. $y=x$

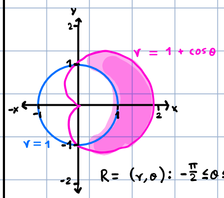
R: $0\le x\le 5$, $0\le y\le x$

$$\int_0^{\frac{\pi}{4}}\int_0^{5\sec\theta}r^2(r\sin\theta)^2\ r\,dr\,d\theta=\int_0^5\int_0^x(x^2+y^2)(y)^2\,dy\,dx$$

Find the area of: Cardioid $r=1+\cos\theta$ and a circle $r=1$.

$$A=\iint_R dA$$
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_1^{1+\cos\theta}r\,dr\,d\theta$$
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac12 r^2\right]_1^{1+\cos\theta}d\theta$$
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac12(1+\cos\theta)^2-\frac12\right]d\theta$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta+\frac12\cos^2\theta\,d\theta \qquad x^2+2ax+a^2=(x+a)^2$$
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta+\frac{\cos^2\theta}{2}\,d\theta \qquad \cos^2\theta=\frac{1+\cos 2\theta}{2}$$
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta+\frac14(1+\cos 2\theta)\,d\theta$$
$$\left[\sin\theta+\frac14\left(\theta+\frac12\sin 2\theta\right)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$\left[\sin\frac{\pi}{2}+\frac14\left(\frac{\pi}{2}+\frac12\sin\pi\right)-\sin-\frac{\pi}{2}+\frac14\left(-\frac{\pi}{2}+\frac12\sin-\pi\right)\right]$$
$$\left(1+\frac{\pi}{8}\right)-\left(-1-\frac{\pi}{8}\right)=2+\frac{\pi}{4}$$



$R=(r,\theta):\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2},\ 1\le r\le 1+\cos\theta$

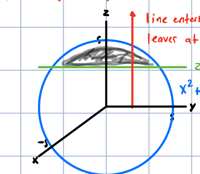
15.5 Triple Integrals in Rectangular Coordinates

$$V=\iiint_D dV \qquad \text{Average value}=\frac{1}{\text{volume of }D}\iiint_D F(x,y,z)\,dV$$

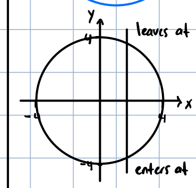
Finding limits of integration (in dz dy dx order)

$x^2+y^2+z^2=25$
$z^2=25-x^2-y^2$
$z=\sqrt{25-x^2-y^2}$

line enters at $z=3$
leaves at $z=\sqrt{25-x^2-y^2}$



$f_1(x,y)=3$
$F_2(x,y)=\sqrt{25-x^2-y^2}$
$x^2+y^2+3^2=25$
$x^2+y^2=25-9=16$



leaves at $y=\sqrt{16-x^2}$
$x^2+y^2=16$
$y^2=16-x^2$
$y=\pm\sqrt{16-x^2}$
enters at $y=-\sqrt{16-x^2}$

$-\sqrt{16-x^2}\le y\le\sqrt{16-x^2}$
$-4\le x\le 4$

$$\text{So, }\iiint_D F(x,y,z)\,dV=\int_{-4}^4\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_3^{\sqrt{25-x^2-y^2}}F(x,y,z)\,dz\,dy\,dx$$

Find average value $\quad \text{Average value}=\frac{1}{\text{volume of }D}\iiint_D F(x,y,z)\,dV$

volume of rectangular solid: $x\cdot y\cdot z$.

$$\frac19\int_0^1\int_0^3\int_0^3(x^2+y^2+z^2)\,dx\,dy\,dz$$
$$\frac19\int_0^1\int_0^3\left[\frac{x^3}{3}+xy^2+xz^2\right]_0^3dy\,dz$$
$$\frac19\int_0^1\int_0^3\left[\frac{3^3}{3}+3y^2+3z^2\right]_0^3dy\,dz$$
$$\frac19\int_0^1\int_0^3(9+3y^2+3z^2)\,dy\,dz$$
$$\frac19\int_0^1\left[9y+y^3+3yz^2\right]_0^3dz$$
$$\frac19\int_0^1\left[9\cdot3+3^3+3\cdot3z^2\right]_0^3dz$$

$$\frac19\int_0^1(27+27+9z^2)\,dz$$
$$\frac19\int_0^1(54+9z^2)\,dz$$
$$\frac19\left[54z+3z^3\right]_0^1$$
$$\frac19\left[54\cdot1+3\cdot1^3\right]_0^1$$
$$\frac19(57)$$
$$\frac{19}{3}$$

15.8 Substitutions in Multiple Integrals

Use the transformation $u=4x+3y$, $v=x+3y$ to evaluate the given integral for the region R bounded by the lines $y=-\frac43x+1$, $y=-\frac43x+4$, $y=-\frac13x$ and $y=-\frac13x+2$.

$$\iint_R f(x,y)\,dx\,dy=\iint_G f(g(u,v),h(u,v))\,|J(u,v)|\,du\,dv$$

$$J(u,v)=\begin{vmatrix}\frac{\partial x}{\partial u}&\frac{\partial x}{\partial v}\\\frac{\partial y}{\partial u}&\frac{\partial y}{\partial v}\end{vmatrix}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

$$\iint_R uv\,|J(u,v)|\,du\,dv$$

$$\iint_R(4x^2+15xy+9y^2)\,dx\,dy=\iint_R(4x+3y)(x+3y)\,dx\,dy$$
write: $u=4x+3y$, $v=x+3y$, $du\,dv$

Since $U=4X+3Y$ $\quad V=X+3Y$ solve for x and y.

$v=x+3y$ $\quad u=4x+3y$ $\quad x=v-3y$
$x=v-3y$ $\quad u=4(v-3y)+3y$ $\quad x=v-3\left(\frac{4v-u}{9}\right)$
$u=4v-12y+3y$ $\quad x=v-\frac{4v+u}{3}$
$u=4v-9y$ $\quad x=\frac{u-v}{3}$
$9y=4v-u$
$y=\frac{4v-u}{9}$

$$|J(u,v)|=\begin{vmatrix}\frac{\partial x}{\partial u}\left[\frac{u-v}{3}\right]&\frac{\partial x}{\partial v}\left[\frac{u-v}{3}\right]\\\frac{\partial y}{\partial u}\left[\frac{4v-u}{9}\right]&\frac{\partial y}{\partial v}\left[\frac{4v-u}{9}\right]\end{vmatrix}=\begin{vmatrix}\frac13&-\frac13\\-\frac19&\frac49\end{vmatrix}$$

$$=\left(\frac13\right)\left(\frac49\right)-\left(-\frac19\right)\left(-\frac13\right)=\frac19$$

therefore,

$$\iint_R (4x+3y)(x+3y)\,dx\,dy = \iint_R uv\,|J(u,v)|\,du\,dv = \frac{1}{9}\iint_G uv\,du\,dv$$

transform the equation $y = -\frac{4}{3}x + 1$

$$\frac{4v-u}{9} = -\frac{4}{3}\left(\frac{u-v}{3}\right) + 1$$

$$9\cdot\frac{4v-u}{9} = -\frac{4u+4v}{9} + 1\cdot 9$$

$$\cancel{4v} - u = -4u + \cancel{4v} + 9$$

$$-9 = -4u + u$$

$$-9 = -3u$$

$$u = 3$$

transform the equation $y = -\frac{4}{3}x + 4$

$$\frac{4v-u}{9} = -\frac{4}{3}\left(\frac{u-v}{3}\right) + 4$$

$$9\cdot\frac{4v-u}{9} = -\frac{4u+4v}{9} + 4\cdot 9$$

$$\cancel{4v} - u = -4u + \cancel{4v} + 36$$

$$-36 = -4u + u$$

$$-36 = -3u$$

$$u = 12$$

transform the equation $y = -\frac{1}{3}x$

$$\frac{4v-u}{9} = -\frac{1}{3}\left(\frac{u-v}{3}\right)$$

$$9\cdot\frac{4v-u}{9} = -\frac{u+v}{9}\cdot 9$$

$$4v - \cancel{u} = -\cancel{u} + v$$

$$3v = 0$$

$$v = 0$$

transform the equation $y = -\frac{1}{3}x + 2$

$$\frac{4v-u}{9} = -\frac{1}{3}\left(\frac{u-v}{3}\right) + 2$$

$$9\cdot\frac{4v-u}{9} = -\frac{u+v}{9} + 2\cdot 9$$

$$4v - \cancel{u} = -\cancel{u} + v + 18$$

$$3v = 18$$

$$v = 6$$

Hence,

$$\frac{1}{9}\iint_G uv\,du\,dv \qquad \frac{1}{9}\int_0^6 \left[\frac{u^2}{2}\right]_3^{12} v\,dv \qquad \frac{15}{2}\left[\frac{v^2}{2}\right]_0^6$$

$$\frac{1}{9}\int_0^6\int_3^{12} uv\,du\,dv \qquad \frac{1}{9}\int_0^6 \left(\frac{135}{2}\right) v\,dv \qquad \frac{15}{2}\left[\frac{v^2}{2}\right]_0^6$$

$$\frac{15}{2}(18) = 135$$

16.1 Line Integrals of Scalar Functions

$\int_C (x+y)\,ds$ over the straight-line segment $x = 3t$, $y = (6-3t)$, $z = 0$ from $(0,6,0)$ to $(6,0,0)$

when
$x = 3t = 0$ $t=0$
$3t = 6$ $t=2$
so, $0 \le t \le 2$

$r(t) \to 3t\,i + (6-3t)\,j + 0k$
$r'(t) \to 3i + 3j$
$|r'(t)| = \sqrt{9+9} = 3\sqrt{2} \to |v(t)|$

so, $\int_C (x+y)\,ds = \int_0^2 (x+y)\,|v(t)|\,dt \qquad ds = |v(t)|$

$$\int_0^2 (\cancel{3t} + 6 - \cancel{3t})\cdot 3\sqrt{2}\,dt = \int_0^2 18\sqrt{2}\,dt = 18\sqrt{2}\,[t]_0^2 = 18\sqrt{2}\,(2-0) = 36\sqrt{2}$$

Integrating over a curve

$\int_C (xy + x + z)\,ds$ along the curve $r(t) = 2t\,i + t\,j + (8-2t)k$, $0 \le t \le 1$.

$r'(t) \to 2i + j - 2k$
$|r'(t)| = \sqrt{4+1+4} = 3dt \to |v(t)| = ds$

$$\int_C f(x,y,z)\,ds = \int_a^b f(g(t),h(t),k(t))\,|v(t)|\,dt$$

$$\int_C (xy+x+z)\,ds = \int_0^1 (2t\cdot t + \cancel{2t} + (8 - \cancel{2t}))\,3dt = \int_0^1 3(2t^2 + 8)\,dt$$

$$3\left[\frac{2}{3}t^3 + 8t\right]_0^1 = 26$$

Integrate $f(x,y,z) = x + \sqrt{y} - z^4$ over the path from (0,0,0) to (1,1,1) given by
C_1: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1$
C_2: $\mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$.

use the equation

$$\int_C (x + \sqrt{y} - z^4)\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$$

For C_1: $r(t) = t\,i + t^2 j$, $0 \le t \le 1$
$x(t) = t$, $y(t) = t^2$, $z(t) = 0$
$x'(t) = 1$, $y'(t) = 2t$, $z'(t) = 0$

$ds = |v'(t)|$
$= \sqrt{1^2 + (2t)^2 + 0^2}\,dt$
$\sqrt{1 + 4t^2}\,dt$

from starting / ending point
from parametrization

$$\int_C (x + \sqrt{y} - z^4)\,ds = \int_0^1 (t + t - 0^4)\sqrt{1+4t^2}\,dt$$

$$\int_0^1 2t\sqrt{1+4t^2}\,dt$$

$u = 1 + 4t^2$
$du = 8t\,dt$
$t\,dt = \frac{1}{8}du$

$$\therefore \int t\frac{\sqrt{u}}{8}\,du = \int \frac{\sqrt{u}}{8}\,du = \frac{u^{\frac{1}{2}+1}}{8\cdot\frac{1}{2}+1} = \frac{u^{\frac{3}{2}}}{12}$$

$$\frac{(1+4t^2)^{3/2}}{12}$$

$$2\left[\frac{(1+4t^2)^{3/2}}{12}\right]$$

$$\frac{1}{6}\left[(1+4t^2)^{3/2}\right]_0^1 = \frac{1}{6}(5)^{\frac{3}{2}} - 1 = \frac{1}{6}(5\sqrt{5} - 1) \quad ①$$

For C_2: $r(t) = i + j + t\,k$, $0 \le t \le 1$
$x(t) = 1$, $y(t) = 1$, $z(t) = t$
$x'(t) = 0$, $y'(t) = 0$, $z'(t) = 1$

$ds = |v'(t)|$
$= \sqrt{1^2}\,dt$
$= 1\,dt$

$$\int_C (x + \sqrt{y} - z^4)\,ds = \int_0^1 (1 + 1 - t^4)\,dt = \int_0^1 (2 - t^4)\,dt$$

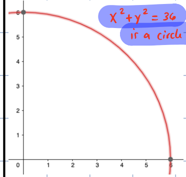
$$(2 - t^4)\,dt = \left[2t - \frac{t^5}{5}\right]_0^1 = \frac{9}{5} \quad ②$$

thus, $\int_C (x + \sqrt{y} - z^4)\,ds$

$$= \frac{1}{6}(5\sqrt{5} - 1) + \frac{9}{5} = \frac{5\sqrt{5}}{6} - \frac{1}{6} + \frac{9}{5} = \frac{5\sqrt{5}}{6} + \frac{49}{30}$$

ANS.

Integrate $F(x,y) = x + y$ over the curve $C: x^2 + y^2 = 36$ in the first quadrant from $(6,0)$ to $(0,6)$.



$x^2 + y^2 = 36 \qquad 0 \le t \le \frac{\pi}{2}$

$r(t) = (6\cos t)i + (6\sin t)j$
$v(t) = (-6\sin t)i + (6\cos t)j$
$|v(t)| = \sqrt{(-6\sin t)^2 + (6\cos t)^2}$
$= \sqrt{36\sin^2 t + 36\cos^2 t}$
$= \sqrt{36(\sin^2 t + \cos^2 t)}$
$= 6\,dt$

$$\int_C F(x,y)\,ds = \int_0^{\frac{\pi}{2}} F(6\cos t, 6\sin t)\cdot 6\,dt$$

$$\int_0^{\frac{\pi}{2}} (6\cos t + 6\sin t)\cdot 6\,dt$$

$$\int_0^{\frac{\pi}{2}} 6(\cos t + \sin t)\cdot 6\,dt$$

$$36\int_0^{\frac{\pi}{2}} (\cos t + \sin t)\,dt$$

$$36\left[\sin t - \cos t\right]_0^{\frac{\pi}{2}}$$

$$36\left[\sin\frac{\pi}{2} - \cos\frac{\pi}{2} - (\sin 0 - \cos 0)\right]$$

$$36\left[1 - 0 - (0 - 1)\right] = 36\cdot 2 = 72$$

Find the mass of a wire that lies along the curve "curve" $r(t) = (t^2 - 5)j + 2t\,k$ $0 \le t \le 3$, if the density is $\delta = \frac{3}{2}t$.

$r(t) \to (t^2 - 5)j + 2t\,k$
$r'(t) \to 2t\,j + 2k$
$|r'(t)| = \sqrt{4t^2+4} = 2\sqrt{t^2+1}\,dt \to |v(t)|$

$$M = \int_C \delta(x,y,z)\,ds = \int_C \delta(x,y,z)\,|v(t)|\,dt$$

$$M = \int_0^3 \frac{3}{\cancel{2}}t\cdot\cancel{2}\sqrt{t^2+1}\,dt = \int_0^3 3t\sqrt{t^2+1}\,dt$$

$u\ (t^2+1)^{\frac{1}{2}}$
$du\ \frac{1}{2}(t^2+1)^{-\frac{1}{2}}\cdot 2t$
$du\ \frac{\cancel{2}t}{\cancel{2}(t^2+1)^{\frac{1}{2}}} = \frac{t}{\sqrt{t^2+1}}$

$t=3 \quad u=\sqrt{10}$
$t=1 \quad u=1$

$t = \sqrt{t^2+1}\,du$
$t = u\,du$

$$= 3\int_1^{\sqrt{10}} u\cdot u\,du = 3\int_1^{\sqrt{10}} u^2\,du$$

$$= \left[\frac{u^3}{3}\right] = \frac{(\sqrt{10})^3}{3} - \frac{1}{3} = \frac{10^{\frac{1}{2}\cdot 3}}{3} - \frac{1}{3}$$

$$= \frac{10^{\frac{1}{2}\cdot 3}}{3} - \frac{1}{3} = \frac{10^{\frac{3}{2}}}{3} - \frac{1}{3} = \frac{10\sqrt{10} - 1}{\cancel{3}}\cdot\cancel{3}$$

$$= 10\sqrt{10} - 1$$

Reference

Adjust integral boundaries

$u = 3y$ $\quad$ $y = 1 \rightarrow u = 3$

$du = 3\,dy$ $\quad$ $y = \ln 5 \rightarrow u = 3\ln 5$

$dy = \frac{1}{3}\,du$

$$\int e^y dy = e^y$$

$$e^1 = e$$

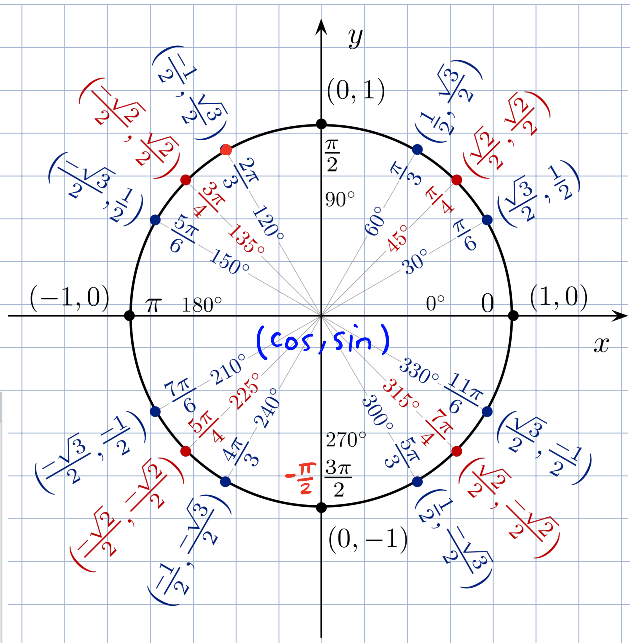
$$e^{x+y} = e^x \cdot e^y$$

$$\int \frac{1}{u} du = \ln(u)$$

$$a^{\log_a(b)} = b$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$



Degree	Radians	$sin\ \theta$	$cos\ \theta$	tan θ	$cosec\ \theta$	$sec\ \theta$	$cot\ \theta$
0°	0	0	1	0	-	1	-
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$ $\frac{1}{\sqrt{3}}$	2	$\frac{2\sqrt{3}}{2}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{2}$	2	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	-	1	-	0
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	-2	$-\frac{\sqrt{3}}{3}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	2	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$
180°	π	0	-1	0	-	-1	-
270°	$\frac{3\pi}{2}$	-1	0	-	-1	-	0
360°	2π	0	1	0	-	1	-

COMMON FACTORING EXAMPLES
$x^2 - a^2 = (x+a)(x-a)$
$x^2 + 2ax + a^2 = (x+a)^2$
$x^2 - 2ax + a^2 = (x-a)^2$
$x^2 + (a+b)x + ab = (x+a)(x+b)$
$x^3 + 3ax^2 + 3a^2x + a^3 = (x+a)^3$
$x^3 + a^3 = (x+a)(x^2 - ax + a^2)$
$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$
$x^{2n} - a^{2n} = (x^n - a^n)(x^n + a^n)$

FUNDAMENTAL IDENTITIES

$\csc\theta = \frac{1}{\sin\theta}$ $\quad$ $\sec\theta = \frac{1}{\cos\theta}$ $\quad$ $1 + \tan^2\theta = \sec^2\theta$ $\quad$ $1 + \cot^2\theta = \csc^2\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\quad$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$ $\quad$ $\sin(-\theta) = -\sin\theta$ $\quad$ $\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

$\cot\theta = \frac{1}{\tan\theta}$ $\quad$ $\sin^2\theta + \cos^2\theta = 1$

DOUBLE ANGLE FORMULAS

$\sin 2x = 2\sin x + \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

Differentiation Rules	
Constant Rule	$\frac{d}{dx}[c] = 0$
Power Rule	$\frac{d}{dx}x^n = nx^{n-1}$
Product Rule	$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
Quotient Rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$
Chain Rule	$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

$\csc\theta = \frac{1}{\sin\theta}$ $\quad$ $\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\quad$ $\cot\theta = \frac{1}{\tan\theta}$

$\sec\theta = \frac{1}{\cos\theta}$ $\quad$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$ $\quad$ $\sin^2\theta + \cos^2\theta = 1$

Derivative	Integral (Antiderivative)
$\frac{d}{dx}n = 0$	$\int 0\,dx = C$
$\frac{d}{dx}x = 1$	$\int 1\,dx = x + C$
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
$\frac{d}{dx}e^x = e^x$ $\quad$ $\frac{d}{dx}e^u = e^u \cdot u'$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}\ln x = \frac{1}{x}$ $\quad$ $\frac{d}{dx}\ln u = \frac{u'}{u}$	$\int \frac{1}{x}\,dx = \ln x + C$
$\frac{d}{dx}n^x = n^x \ln x$	$\int n^x\,dx = \frac{n^x}{\ln n} + C$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\int \tan x \sec x\,dx = \sec x + C$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\int \cot x \csc x\,dx = -\csc x + C$
$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$
$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$	$\int -\frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + C$
$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \arctan x + C$
$\frac{d}{dx}\text{arccot}\, x = -\frac{1}{1+x^2}$	$\int -\frac{1}{1+x^2}\,dx = \text{arccot}\, x + C$
$\frac{d}{dx}\text{arcsec}\, x = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \text{arcsec}\, x + C$
$\frac{d}{dx}\text{arccsc}\, x = -\frac{1}{x\sqrt{x^2-1}}$	$\int -\frac{1}{x\sqrt{x^2-1}}\,dx = \text{arccsc}\, x + C$